Grade 9/10 Math Circles
April 3, 2024
Probability III

Last time, we discussed independence, conditional probability, and Bayes' Theorem. Today, we will use these ideas to solve the False Negative and Monty Hall problems. probability theory.

Warm-Up Problems

Let's start by reviewing independence, conditional probability, and Bayes' Theorem.

Recall that independent events follow the rule $P(A \cap B) = P(A) \cdot P(B)$.

Example 1. Suppose you create a PIN by randomly selecting 3 numbers from 0-9. What is the probability it starts with a 1 and ends with a 9?

Solution: Let O mean the PIN starts with 1 and N mean the PIN ends with 9. There are 10 possibilities for each number, each equally likely, so $P(O) = P(N) = 1/10$.

Each number is chosen separately, so they are independent. So,

$$P(O \cap N) = P(O) \cdot P(N) = \frac{1}{10} \cdot \frac{1}{10} = \frac{1}{100}$$

Exercise 1. How does the answer to example 1 change if the PIN is 6 digits long?

Recall that when working with dependent events, the notion of conditional probability is very useful. The conditional probability of A given B is the probability that A occurs, given that B already happened. The formula is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

We also rearranged this formula to solve for $P(A \cap B)$ when the events were dependent. That is, $P(A \cap B) = P(A|B) \cdot P(B)$.

Example 2. Suppose you flip a coin 3 times. What is the probability that you flip heads exactly once, given that you flip heads at least once?

Let A be the event that you flip heads exactly once and B be the event that you flip heads at least once. Notice that A is a subset of B - if you flip heads exactly once then you certainly flipped heads at least once. So, $A \cap B = A$.

There are 8 possible outcomes of flipping a coin 3 times, all equally likely.

There are 3 ways to flip exactly one head (HTT, THT, TTH), so $P(A) = 3/8$.

There is only 1 way not to flip any heads (TTT), so $P(B) = 1 - P\left(B^C\right) = 1 - 1/8 = 7/8$.

Putting this together with the definition of conditional probability,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{3/8}{7/8} = \frac{3}{7}$$

Exercise 2. Suppose you flip a coin 3 times. What is the probability that you flip heads exactly two times, given that you flip heads at least once?

Finally, we learned how to calculate the probability of one event based on the probability that another event occurs. The formula for this, using the rearrangement of conditional probability to replace the intersections, is

$$P(B) = P(B \cap A) + P\left(B \cap A^C\right) = P(B|A) \cdot P(A) + P\left(B|A^C\right) \cdot P\left(A^C\right)$$

We combined this with Bayes' Theorem, which gives us the relationship between $P(A|B)$ and $P(B|A)$:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Example 3. Suppose there are two dice: a regular die and a weighted die. The probability of the weighted die rolling 6 is 25% and the probability of rolling each other number is 15%. You randomly select a die and roll it.

1. What is the probability you do **not** roll a 6?
2. If you do **no**t roll a 6, what is the probability that it is weighted?

Solution: Let W mean the die is weighted and S mean you roll a 6. Note that there is a 50% chance of choosing the weighted die, so $P(W) = P\left(W^C\right) = 0.5$.

1. The probability that you roll a 6 depends on whether or not the die is weighted. So, we will split the probability based on that.

$$P\left(S\right) = P(S|W) \cdot P(W) + P\left(S|W^C\right) \cdot P\left(W^C\right) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{2} = \frac{5}{24}$$

   Using the complement rule, we get that

$$P\left(S^C\right) = 1 - P(S) = 1 - \frac{5}{24} = \frac{19}{24}$$

2. We want to find $P\left(W|S^C\right)$. Using Bayes' Theorem,

$$P\left(W|S^C\right) = \frac{P\left(S^C|W\right) \cdot P(W)}{P\left(S^C\right)} = \frac{\frac{3}{4} \cdot \frac{1}{2}}{\frac{19}{24}} = \frac{9}{19}$$

   (Note that $P\left(S^C|W\right)$ is the probability of not rolling a six on the weighted die, which is just $1 - P(S|W) = 0.75$.)

Exercise 3. Men are surprisingly likely to be colour-blind. About 1/12 of men are colour-blind while only 1/200 of other people are colour-blind. Suppose that men make up 50% of the population.

1. What is the probability that a random person is colour-blind?
2. Given that someone is colour-blind, what is the probability that they are a man?

False Negative Problem

Example 4. Suppose that during the pandemic, 2% of the population was sick with COVID. A COVID test has a 1% chance of giving you a false positive (positive result when you are healthy) and a 10% chance of giving you a false negative (negative result when you are sick). What is the probability that you are sick, given that your test result was negative?

Let S mean you are sick, H mean you are healthy, N mean you had a negative test, and P mean you had a positive test. Note that being sick and being healthy are complements, as are positive and negative tests. A false negative would be represented as $N|S$ and a false positive would be represented as $P|H$. We want to find $P(S|N)$.

Using Bayes' Theorem,

$$P(S|N) = \frac{P(N|S) \cdot P(S)}{P(N)}$$

We know that $P(N|S) = 0.1$ and $P(S) = 0.02$.

To find the probability of a negative test, we need to consider two cases: you are sick with COVID or you are not sick with COVID. This is the same technique we used previously when splitting up the probabilities of one event based on another event:

$$P(N) = P(N|S) \cdot P(S) + P(N|H) \cdot P(H)$$

The probability of being healthy can be found with the complement rule: $P(H) = 1 - P(S) = 0.98$.

We also need the probability of an accurate negative, $N|H$. A healthy person who takes a COVID test will either get a positive or negative result, so $P(N|H) = 1 - P(P|H) = 0.99$.

Putting this all together gives:

$$\begin{aligned} P(S|N) &= \frac{P(N|S) \cdot P(S)}{P(N)} \\ &= \frac{P(N|S) \cdot P(S)}{P(N|S) \cdot P(S) + P(N|H) \cdot P(H)} \\ &= \frac{0.1 \cdot 0.02}{0.1 \cdot 0.02 + 0.99 \cdot 0.98} \\ &\approx 0.002 \end{aligned}$$

You might be surprised that this value is much less than the false negative rate. The reason for the difference is that the vast majority of people who take tests are not sick, so most negative results will be accurate.

Exercise 4. Continuing with the setup from the false negative problem, what is the probability that you are healthy, given that your test result was positive?

Monty Hall Problem

Example 5. You are on a game show. There are three doors; one has a sports car behind it and the other two have goats. You get to keep whatever is behind the door that you choose.

After you make your selection, the host randomly opens one of the doors you did not choose to reveal a goat. You can then decide to switch to the other closed door or stick with your original choice. Does switching doors give you a higher probability of winning the sports car?

Solution: We will find the probability of winning for each strategy. Let's say the door that you choose is A, the door that the host opens is B, and the door left unopened is C.

When you first make your selection, you have no additional information about what is the behind the doors. So, the probability the car is behind door A is 1/3.

Once the host opens door B, you have some more information - you know that the car is not behind door B. We want to know the probability the car is behind door C given that the host opens door B.

$$P(C|\text{opens B}) = \frac{P(\text{opens B}|C) \cdot P(C)}{P(\text{opens B})}$$

Since the goats and cars were placed randomly at the start, $P(C) = 1/3$.

$P(\text{opens B}|C)$ is the likelihood that the host opens door B given that the car is behind door C. The host cannot open door A since you chose it, and they also cannot choose door C since it has the car. So, the host must choose open door B, meaning $P(\text{opens B}|C) = 1$.

Finally, the probability that the host opens door B depends on which door the car is behind. If we split up the probability by which door the goat is behind, we get:

$$P(\text{opens B}) = P(\text{opens B}|A) \cdot P(A) + P(\text{opens B}|B) \cdot P(B) + P(\text{opens B}|C) \cdot P(C)$$

If the car is behind door A, the host can safely open either door, so $P(\text{opens B}|A) = 1/2$.

As the host will never show the car, $P(\text{opens B}|B) = 0$.

So,

$$P(\text{opens B}) = \frac{1}{2} \cdot \frac{1}{3} + 0 + 1 \cdot \frac{1}{3} = \frac{1}{2}$$

Plugging all our values into Bayes' Theorem gives

$$P(C|\text{opens B}) = \frac{P(\text{opens B}|C) \cdot P(C)}{P(\text{opens B})} = \frac{1 \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

Therefore, you should always switch doors!

Random Variables and Expected Values

So far, we have discussed the probability of individual events from a sample space. It is also useful to think about what the most likely outcome of an experiment is, especially when the outcomes have numerical values. We can do this by thinking about the outcomes of an experiment as a random variable.

Definition 1. A *random variable* is a variable which takes on different numerical values with different probabilities. Specifically, the values are the outcomes of an experiment and their probabilities are the likelihoods of them occurring in the experiment.

Let's first look at some examples based on experiments we have discussed in the past.

Example 6. Rolling a die can be represented by a random variable X which takes on the values $\{1, 2, 3, 4, 5, 6\}$, each with probability $1/6$.

We use the notation $P(X = i) = k$ to mean that the probability that $X = i$ is k.

Example 7. The number of heads from flipping 2 coins is a random variable X with

- $P(X = 0) = 1/4$
- $P(X = 1) = 1/2$
- $P(X = 2) = 1/4$

Exercise 5. Determine if each of the following are a random variable:

1. The number of face cards in a hand of 4 cards.
2. The winner of a chess tournament.
3. The number of games a player in a chess tournament wins.

We can use random variables to calculate the expected value of an experiment. This value is often referred to as the average value.

Definition 2. The *expected value* of a random variable X which has possible values v_1 to v_n is

$$E[X] = v_1 \cdot P(X = v_1) + \ldots + v_n \cdot P(X = v_n)$$

Example 8. Find the expected value of rolling a die.

Solution: Let X be the outcome of rolling a die. The values X can take are $\{1, 2, 3, 4, 5, 6\}$, and each has probability $1/6$.

$$\begin{aligned} E[X] &= 1 \cdot P(X = 1) + \ldots + 6 \cdot P(X = 6) \\ &= 1 \cdot \frac{1}{6} + \ldots + 6 \cdot \frac{1}{6} \\ &= (1 + 2 + 3 + 4 + 5 + 6) \cdot \frac{1}{6} \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

So, the expected value of a die roll is 3.5, even though we can never roll it! This is because we are getting an average, rather than a most likely outcome.

We can use expected value calculations to determine how much money you can expect to win from a game.

Example 9. Your friend brings a deck of cards to school. They say that they will pay you \$1 if you draw a face card, but you have to pay them \$0.50 if you draw an odd number. Should you play the game?

Solution: Let X be the value of the card you draw, from your perspective.

Notice that there are 3 face cards and 5 odd numbered cards per suit, for a total of 12 face cards and 20 odd numbered cards in the deck. So, $P(X = 1) = 12/52$ and $P(X = -0.5) = 20/52$.

$$\begin{aligned} E[X] &= 1 \cdot P(X = 1) - 0.5 \cdot P(X = -0.5) \\ &= 1 \cdot \frac{12}{52} - 0.5 \cdot \frac{20}{52} \\ &= \frac{12}{52} - \frac{10}{52} = \frac{2}{52} \end{aligned}$$

You would expect to make a few cents by playing this game, so you should play it!

Exercise 6. Find the expected number of heads if you flip 2 coins.

Exercise 7. Your friend suggests that you bet on the outcome of a die roll. If the roll is 1 through 4, they will pay you \$3. If the roll is 5 or 6, you will pay them \$$k$. If $k = 7$, should you play the game?

Bonus: What value of k would make the game fair? (Neither you nor your friend expect to make money.)